The Elliptic Cylinder Function of Class K.

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The object of this paper is the synthetic treatment of the Elliptic Function of Class K and the computation of tables of values that may be useful to physicists in studying the properties of elliptical membranes and elliptical cylinders. The literature on this subject is very limited, as the only serious attempts to discuss this function have been made by Mathieu and Heine. The investigation of Mathieu in his Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique (*Jour. de Liouville*) made some progress in the theoretical treatment of the function, but did not lead to definite conclusions. Heinrich Weber, in *Annal. von Clebsch und Neumann*, Bd. I, says of Mathieu's work: "Die Integration ist dort durch Reihen bewerkstelligt, von denen mit grossem Fleisse eine beträchtliche Anzahl Glieder berechnet sind, für welche aber ebenfalls kein allgemeines Gesetz angegeben ist. Diese Untersuchungen mögen daher für den Physiker immerhin von grossem Werte sein, mathematisch scheint mir das Problem dadurch der Lösung wenig näher gebracht zu sein, als durch die Aufstellung der gewöhnlichen Differentialgleichung selbst."

E. Heine, in *Kugelfunktionen*, Bd. I, II, makes greater progress in the analytic treatment, but does not give a satisfactory proof* of the convergence of the series and does not carry the investigation far enough to make the results useful to the physicist.

Laplace's equation $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$ is reduced by Heine, after various transformations and the introduction of Lamé's elliptical coordinates, to the convenient form

$$\frac{d^2 E}{d\phi^2}+\left(\frac{8}{b}\cos 2\phi+4z\right)E=0. \tag{1}$$

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*Since the above statement was written, a satisfactory proof has appeared in the Inaugural Dissertation of Simon Dannacher, Zürich, 1906.

Our investigation will be limited to the function of the Class K,

$$E = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\phi, \tag{2}$$

in which a is independent of ϕ but a function of z and of the argument b. Under what conditions is this a solution of (1)?

Substituting this value of E in (1), we find the following recursion formulas, showing the relation between the coefficients:

$$\begin{aligned} a_1 &= -\tfrac{1}{2} bza_0, \\ a_2 &= b(1-z)a_1 - a_0, \\ a_3 &= b(4-z)a_2 - a_1, \\ &\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots \\ a_{n+1} &= b(n^2-z)a_n - a_{n-1}. \end{aligned} \tag{3}$$

To determine the necessary conditions for the convergence of (2), it is necessary first to show that $\lim_{n=\infty} a_n = 0$.

Substituting a finite portion of (2) in (1) gives

$$\frac{d^2E}{d\phi^2} + \left(\frac{8}{b}\cos 2\phi + 4z\right)E = \frac{4}{b}\left(a_n \cos(2n+2)\phi - a_{n-1}\cos 2n\phi\right). \tag{4}$$

If values of z can be found that will make a_n and a_{n-1} arbitrarily small, the first member may be made to approximate zero.

Assuming that $a_0 = 1$, a_1, a_2, ..., a_n are rational integral functions of z, and equations (3), in the form $a_n = f(z) = 0$, can be solved for various values of the argument b. Equations (3) give the following values of a_1, a_2, ..., a_7, explicitly in terms of b and z. Placing these equal to zero we compute the values of z that will make a_1, ..., a_7 approximately zero.

$$\begin{aligned} a_1 &= -\frac{1}{2}bz = 0, \\ a_2 &= +\frac{1}{2}b^2\left(z^2 - z - \frac{2}{b^2}\right) = 0, \\ a_3 &= -\frac{1}{2}b^3\left(z^3 - 5z^2 + \left(4 - \frac{3}{b^2}\right)z + \frac{8}{b^2}\right) = 0, \\ a_4 &= +\frac{1}{2}b^4\left(z^4 - 14z^3 + \left(49 - \frac{4}{b^2}\right)z^2 + \left(-36 + \frac{36}{b^2}\right)z + \left(-\frac{72}{b^2} + \frac{2}{b^4}\right)\right) = 0, \end{aligned} \tag{5}$$

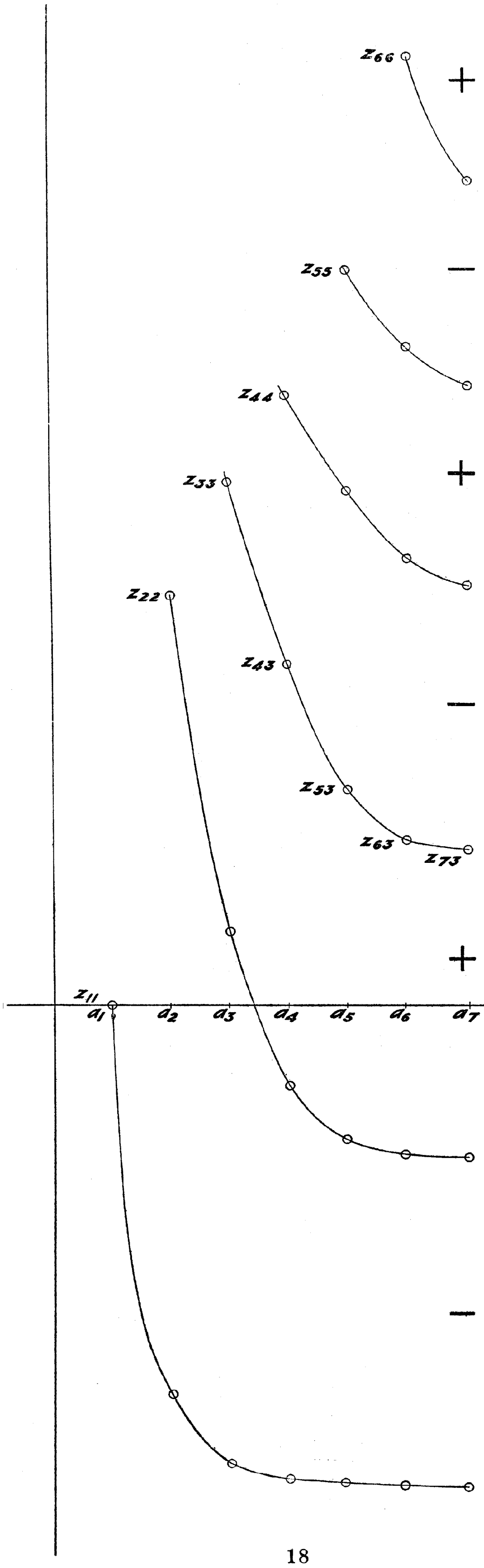

FIG. 1

$a_n = 0$

$n = 1, 2, --, 7$

$b = 0.1$

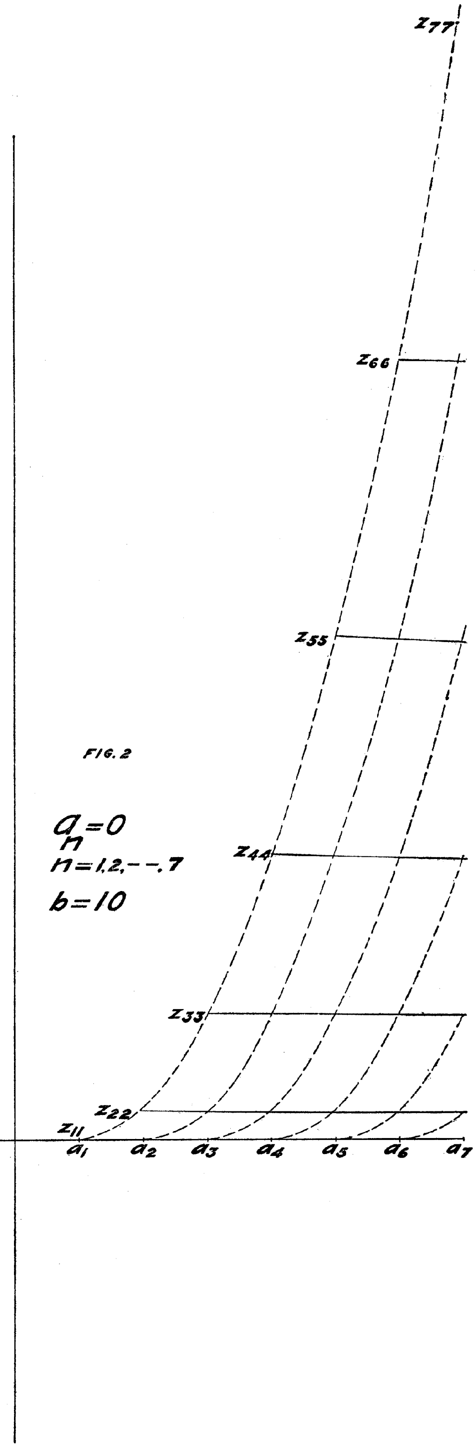

FIG. 2

$a_n = 0$

$n = 1, 2, --, 7$

$b = 10$

$$a_5 = -\frac{1}{2}b^5\left(z^5 - 30z^4 + \left(273 - \frac{5}{b^2}\right)z^3 + \left(-820 + \frac{105}{b^2}\right)z^2 + \left(576 - \frac{652}{b^2} + \frac{5}{b^4}\right)z + \left(\frac{1152}{b^2} - \frac{40}{b^4}\right)\right) = 0,$$

$$a_6 = +\frac{1}{2}b^6\left(z^6 - 55z^5 + \left(1023 - \frac{6}{b^2}\right)z^4 + \left(-7645 + \frac{244}{b^2}\right)z^3 + \left(21076 - \frac{3326}{b^2} + \frac{9}{b^4}\right)z^2 + \left(-14400 + \frac{17488}{b^2} - \frac{201}{b^4}\right)z + \left(-\frac{28800}{b^2} + \frac{1072}{b^4} - \frac{2}{b^6}\right)\right) = 0,$$

$$a_7 = -\frac{1}{2}b^7\left(z^7 - 91z^6 + \left(3003 - \frac{7}{b^2}\right)z^5 + \left(-44473 + \frac{490}{b^2}\right)z^4 + \left(296296 - \frac{12383}{b^2} + \frac{14}{b^4}\right)z^3 + \left(-773136 + \frac{138044}{b^2} - \frac{630}{b^4}\right)z^2 + \left(578400 - \frac{658944}{b^2} + \frac{8960}{b^4} - \frac{7}{b^6}\right)z + \left(\frac{1036800}{b^2} - \frac{39744}{b^4} + \frac{112}{b^6}\right)\right) = 0.$$

In order to study these equations and to deduce from them properties of the equation $a_\infty = 0$, we compute tables of values showing the roots to four decimal places and the values of the functions $a_1, a_2, \ldots, a_7$ for these approximate roots. The Tables I, II, III are given for $b = 0.1$, $b = 10$ and $b = 100$. The last figure of every result is given exactly as found, and is not increased by unity if the next figure is five or more.

Tables I, II, III and Figs. 1, 2 show how rapidly these roots approach a constant limiting value as n increases without limit. This line of argument forces us to use a_∞ as a function of z of infinite degree and to treat $a_\infty = 0$ as an equation of infinite degree. This extension of the function concept may be justified by the same necessity which forces us in certain problems to use infinity as a limit.

From equations (5) we deduce the following:

$$b = 0.1.$$

$$(6)\quad a_1 = -\frac{0.1}{2}(z) = 0,$$

$$a_2 = +\frac{(0.1)^2}{2}(z^2 - z - 200) = 0,$$

$$a_3 = -\frac{(0.1)^3}{2}(z^3 - 5z^2 - 296z + 800) = 0,$$

$$a_4 = + \frac{(0.1)^4}{2} (z^4 - 14z^3 - 351z^2 + 3564z + 12800) = 0,$$

$$a_5 = - \frac{(0.1)^5}{2} (z^5 - 30z^4 - 227z^3 + 9680z^2 - 14624z - 284800) = 0,$$

$$a_6 = + \frac{(0.1)^6}{2} (z^6 - 55z^5 + 423z^4 + 16755z^3 - 221524z^2 - 275600z + 5840000) = 0,$$

$$a_7 = - \frac{(0.1)^7}{2} (z^7 - 91z^6 + 2303z^5 + 4527z^4 - 802004z^3 + 6731264z^2 + 17224000z - 181760000) = 0.$$

$$b = 10.$$

(7) $$a_1 = - \frac{10}{2} (z) = 0,$$

$$a_2 = + \frac{10^2}{2} (z^2 - z - 0.02) = 0,$$

$$a_3 = - \frac{10^3}{2} (z^3 - 5z^2 + 3.97z + 0.08) = 0,$$

$$a_4 = + \frac{10^4}{2} (z^4 - 14z^3 + 48.96z^2 - 35.64z - 0.7198) = 0,$$

$$a_5 = - \frac{10^5}{2} (z^5 - 30z^4 + 272.95z^3 - 818.95z^2 + 569.4805z + 11.516) = 0,$$

$$a_6 = + \frac{10^6}{2} (z^6 - 55z^5 + 1022.94z^4 - 7642.56z^3 + 21042.7409z^2 - 14225.1401z - 287.892802) = 0,$$

$$a_7 = - \frac{10^7}{2} (z^7 - 91z^6 + 3002.93z^5 - 44468.1z^4 + 296172.1714z^3 - 771755.623z^2 + 511811.455993z + 10364.025712) = 0.$$

$$b = 100.$$

(8) $$a_1 = - \frac{10^2}{2} (z) = 0,$$

$$a_2 = + \frac{10^4}{2} (z^2 - z - 0.0002) = 0,$$

$$a_3 = - \frac{10^6}{2} (z^3 - 5z^2 + 3.9997z + 0.0008) = 0,$$

$$a_4 = + \frac{10^8}{2} (z^4 - 14z^3 + 48.9996z^2 - 35.9964z - 0.00719998) = 0,$$

$$a_5 = - \frac{10^{10}}{2} (z^5 - 30z^4 + 272.9995z^3 - 819.9895z^2 + 575.93480005z + 0.1151996) = 0,$$

$$a_6 = + \frac{10^{12}}{2}(z^6 - 55z^5 + 1022.9994z^4 - 7644.9756z^3 + 21075.66740009z^2 - 14398.25120201z - 2.879989280002) = 0,$$

$$a_7 = - \frac{10^{14}}{2}(z^7 - 91z^6 + 3002.9993z^5 - 44472.951z^4 + 296294.76170014z^3 - 773122.1956063z^2 + 518334.105689599993z + 103.679602560112) = 0.$$

These equations (6), (7), (8) are solved by Horner's method, and the last remainders are multiplied by the factors before the parentheses in the equations under consideration. By z_{63} we represent the third root of $a_6 = 0$, and by a_{63} the value of a_6 for this approximate z_{63}. See Tables I, II, III.

MAXIMUM AND MINIMUM VALUES OF a_∞.

The maxima and minima values of a_n can not be found by the usual methods when n increases beyond all limits; since a_∞ can not be expressed explicitly in terms of z, and $a'_\infty = 0$ is an equation of an infinite degree.

Substituting $a_n = b^n (n-1)!^2 \beta_n$,* we compute the values of $\bar{z}$ in $\beta'_n = 0$ for finite values of n, $\bar{z}_{ni}$ being taken graphically as the abscissa corresponding to the maximum or minimum value of β_n between the i^{th} and $i+1^{th}$ roots of $\beta_n = 0$. We compute $\bar{z}$ from $\beta'_n = 0$ by Horner's method, accurate to one decimal place. Substituting these values of $\bar{z}$ in β_n by Horner's method, we obtain close approximations for $\bar{\beta}_n$.

I. *Argument* $b = 10$.

(9) $$\beta_n = \frac{a_n}{b^n (n-1)!^2}.$$

(10) $$\beta_2 = + \frac{1}{2}(z^2 - z - 0.02),$$

$$\beta_3 = - \frac{1}{8}(z^3 - 5z^2 + 3.97z + 0.08),$$

$$\beta_4 = + \frac{1}{72}(z^4 - 14z^3 + 48.96z^2 - 35.64z - 0.7198),$$

$$\beta_5 = - \frac{1}{1152}(z^5 - 30z^4 + 272.95z^3 - 818.95z^2 + 569.4805z + 11.516),$$

* See Dannacher, p. 16.

$$\beta_6 = +\frac{1}{28800}(z^6 - 55z^5 + 1022.94z^4 - 7642.56z^3 + 21042.7409z^2 - 14225.1401z - 287.892802),$$

$$\beta_7 = -\frac{1}{103600}(z^7 - 91z^6 + 3002.93z^5 - 44468.1z^4 + 296172.1714z^3 - 771755.623z^2 + 511811.455993z + 10364.025712).$$

From these equations Table V is computed.

Substituting (9) in the recursion formula,

$$a_n = b\left((n-1)^2 - z\right) a_{n-1} - a_{n-2}, \tag{11}$$

$$b^n (n-1)!^2 \beta_n = b\left((n-1)^2 - z\right) b^{n-1} (n-2)!^2 - b^{n-2} (n-3)!^2 \beta_{n-2},$$

$$\beta_n = \left(1 - \frac{z}{(n-1)^2}\right) \beta_{n-1} - \frac{1}{b^2(n-1)^2(n-2)^2} \beta_{n-2}. \tag{12}$$

The first minima for β_n are found by computation to have the relation $|\overline{\beta}_{21}| > |\overline{\beta}_{31}| > |\overline{\beta}_{41}|$, and all are negative. See Table V.

$$\overline{\beta}_{31} = \left(1 - \frac{0.4}{2^2}\right)\beta_{21} - \frac{1}{10^2 \times 2^2 \times 1^2}\beta_{11}, \text{ for } \bar{z}_{31} = +0.4. \tag{13}$$

$$\overline{\beta}_{41} = \left(1 - \frac{0.4}{3^2}\right)\beta_{31} - \frac{1}{10^2 \times 3^2 \times 2^2}\beta_{21}, \text{ for } \bar{z}_{41} = +0.4. \tag{14}$$

$$\overline{\beta}_{51} = \left(1 - \frac{0.4}{4^2}\right)\beta_{41} - \frac{1}{10^2 \times 4^2 \times 3^2}\beta_{31}, \text{ for } \bar{z}_{41} = +0.4. \tag{15}$$

$$\overline{\beta}_{\infty 1} = \left(1 - \frac{\bar{z}_{\infty 1}}{\infty}\right)\beta_{\infty -1, 1} - \frac{1}{10^2 \times \infty^2 \times \infty^2}\beta_{\infty -2, 1}, \text{ for } \bar{z}_{\infty 1}. \tag{16}$$

The first two roots of $\beta_n = 0$ must lie between -2 and 0, $+1$ and $+4$, respectively, as proved by Heine.* Hence the first minimum lies between -2 and $+4$. Beginning with $\overline{\beta}_{51}$, the parenthesis of (15) cannot vary from unity by more than one-fourth, and the parenthesis rapidly approaches the limit unity in $\overline{\beta}_{n1}$ as n increases without limit. The last term in (15) equals $-0.0007\,\beta_{31}$ and in subsequent equations rapidly vanishes, so that the first term in (15) and in subsequent equations controls the sign and $\overline{\beta}_{n1}$ approaches the limit $\overline{\beta}_{n-1,1}$ for $\bar{z}_{n1}$. Hence $|\overline{\beta}_{n1}| < |\overline{\beta}_{n-1,1}| \cdots < |\overline{\beta}_{51}|$ and

$$\lim_{n=\infty} \overline{\beta}_{n1} = \overline{\beta}_{n-1,1}. \tag{17}$$

From computations it is evident that the first minimum of $\beta_\infty = 0$ is numerically less than 0.11 and negative; approximately, -0.09.

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* *Kugelfunktionen*, I, 412.

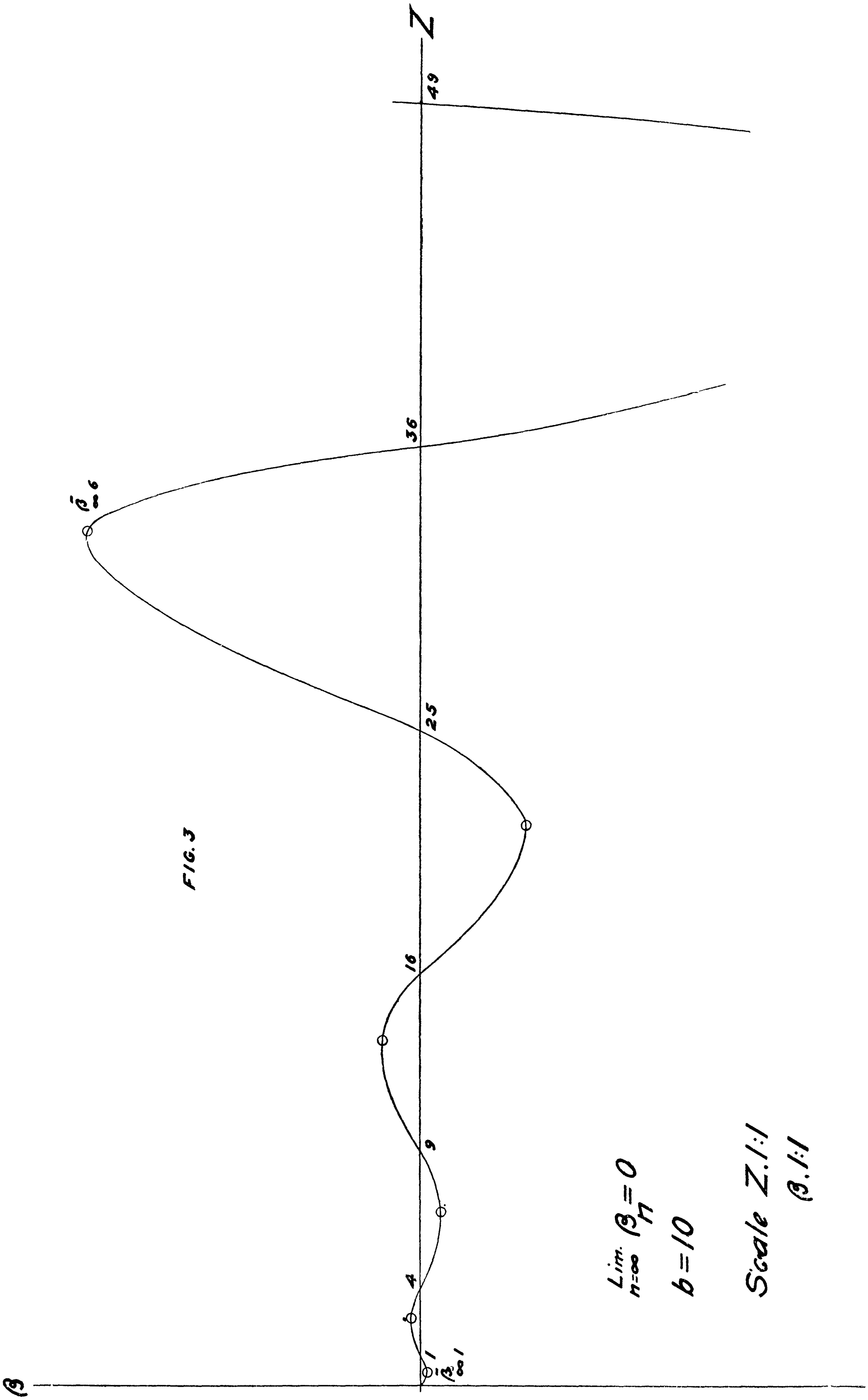


FIG. 3

By similar reasoning, the first maximum $\bar{\beta}_{\infty 2}$ is $+0.33$, approximately. By further computation and reasoning from the recursion formula, it is found that the succeeding minima and maxima of β_n converge to limits for $n=\infty$, each larger than the preceding numerically, and all differing from zero.

The general form of the curve $\beta_\infty = 0$ is shown by Fig. 3, the nullpoints being determined by $z_{\infty 1}, z_{\infty 2}, \ldots, z_{\infty 7}$, as subsequently computed and given in Table II.

II. *Argument* $b = 100$.

For argument $b=100$, $\beta_\infty = 0$ has nullpoints still nearer 0, 1, 4, 9, ..., $(n-1)^2$, as shown in Table III. The locus of $\beta_\infty = 0$, for $b=100$, has the same general form as Fig. 3, the first minimum and the first maximum being nearly the same as for $b=10$. In no case is a maximum or minimum zero.

III. *Argument* $b = 0.1$.

The maxima and minima values of β_∞ for $b=0.1$ present greater difficulties, due to the fact that $(0.1)^2$ occurs in all values of β_n and to the fact that the roots of $\beta_n = 0$ do not fall in the regular intervals -2, 1, 4, 9, ..., $(n-1)^2$, n^2 until

$$b\,(n-2) > 1. \tag{18}$$

See Table I and Heine's *Kugelfunktionen*, I, 407.

To determine the laws governing these maxima and minima values, we compute $\bar{z}$ and $\bar{\beta}$ found in Table IV, using the following equations:

$$\beta_2 = +\frac{1}{2}\left(z^2 - z - \frac{2}{b^2}\right) = +\frac{1}{2}(z^2 - z - 200)\,, \tag{19}$$

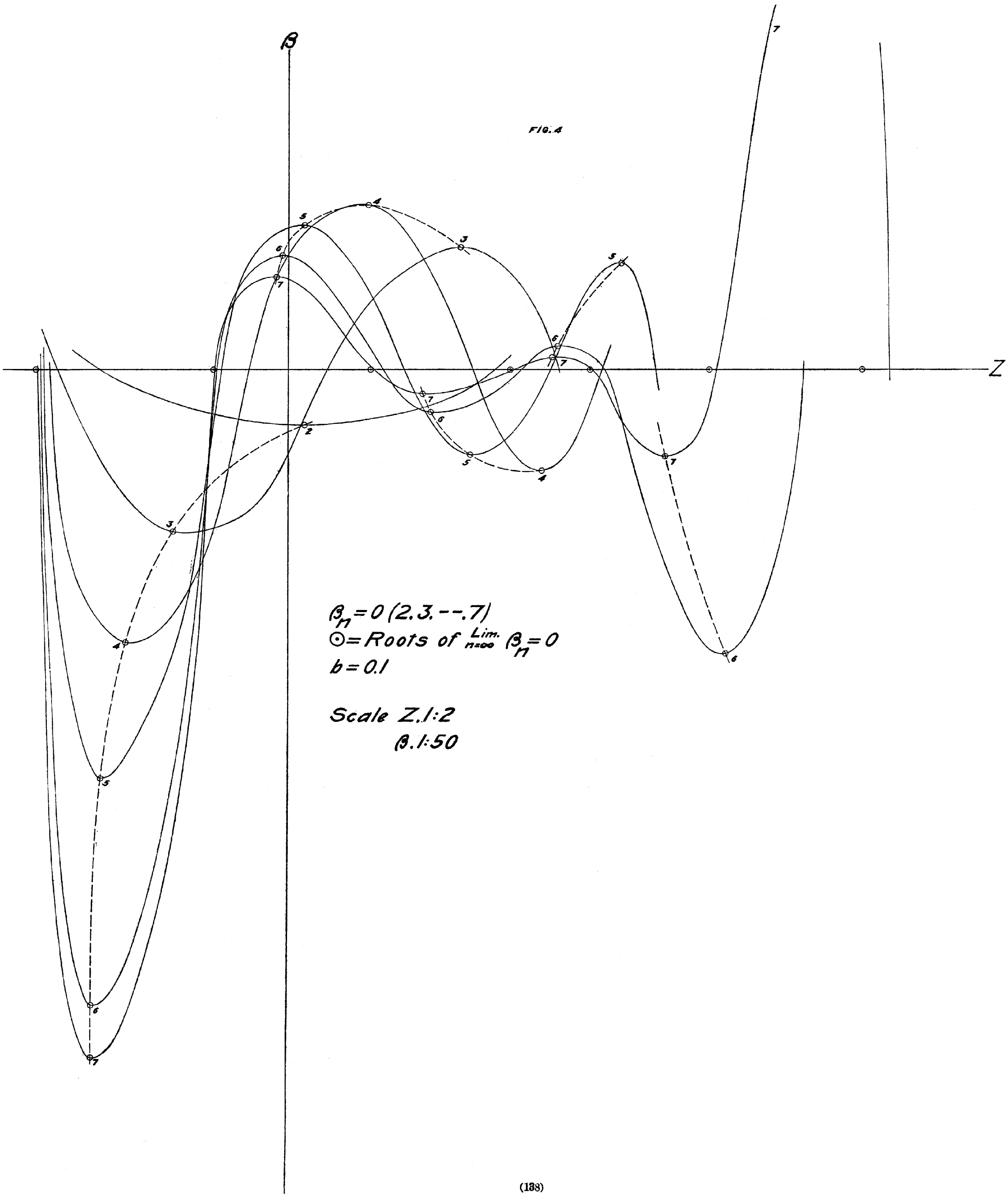
$$\beta_3 = -\frac{1}{8}(z^3 - 5z^2 - 296z + 800)\,, \tag{20}$$

$$\beta_4 = +\frac{1}{72}(z^4 - 14z^3 - 351z^2 + 3564z + 12800)\,, \tag{21}$$

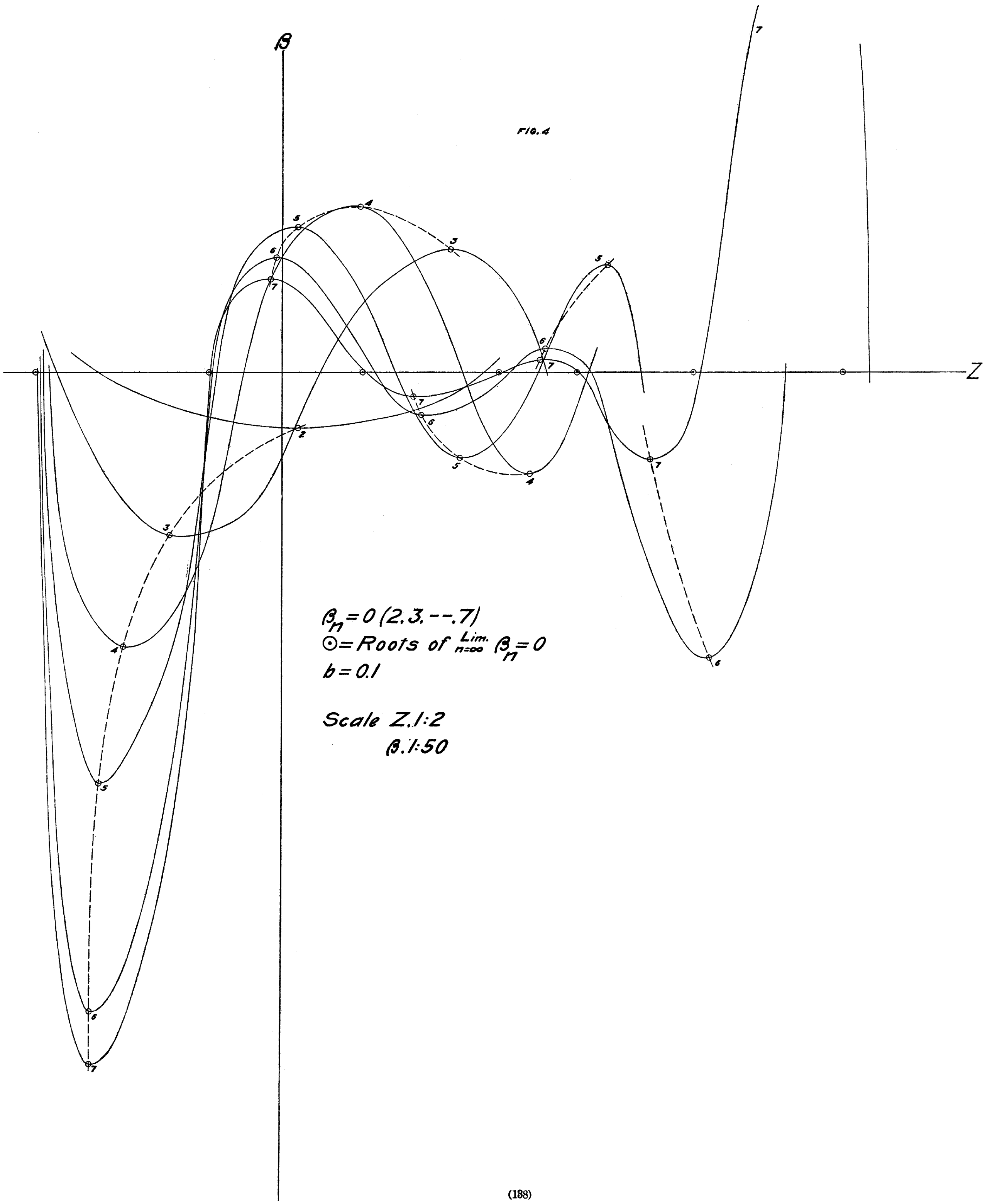
$$\beta_5 = -\frac{1}{1152}(z^5 - 30z^4 - 227z^3 + 9680z^2 - 14624z - 284800)\,, \tag{22}$$

$$\beta_6 = +\frac{1}{28800}(z^6 - 55z^5 + 423z^4 + 16755z^3 - 221524z^2 - 275600z + 5840000)\,, \tag{23}$$

$$\beta_7 = -\frac{1}{1036800}(z^7 - 91z^6 + 2303z^5 + 4527z^4 - 802004z^3 + 6731264z^2 + 17224000z - 181760000)\,, \tag{24}$$



β
Z
FIG. 4
$\beta_n = 0$ (2.3.--.7)
⊙ = Roots of $\lim_{n=\infty} \beta_n = 0$
b = 0.1
Scale Z. 1:2
β. 1:50


β
Z
FIG. 4
$\beta_n = 0\ (2, 3, --, 7)$
$\odot$ = Roots of $\lim_{n=\infty} \beta_n = 0$
$b = 0.1$
Scale Z. 1:2
β. 1:50

(25) $\beta_2' = +1\left(z - \tfrac{1}{2}\right) = 0,$

(26) $\beta_3' = -\dfrac{3}{8}\left(z^2 - 3\tfrac{1}{3}z - 98\tfrac{2}{3}\right) = 0,$

(27) $\beta_4' = +\dfrac{4}{72}\left(z^3 - 10\tfrac{1}{2}z^2 - 175\tfrac{1}{2}z + 891\right) = 0,$

(28) $\beta_5' = -\dfrac{5}{1152}\left(z^4 - 24z^3 - 136\tfrac{1}{5}z^2 + 3872z - 2924\tfrac{4}{5}\right) = 0,$

(29) $\beta_6' = +\dfrac{6}{28800}\left(z^5 - 45\tfrac{5}{6}z^4 + 282z^3 + 8377\tfrac{1}{2}z^2 - 73841\tfrac{1}{3}z - 45933\tfrac{1}{3}\right) = 0,$

(30) $\beta_7' = -\dfrac{7}{1036800}\left(z^6 - 78z^5 + 1645z^4 + 2301\tfrac{1}{7}z^3 - 343716z^2 + 1923218\tfrac{2}{7}z + 2460571\tfrac{3}{7}\right) = 0.$

In Fig. 4, we locate the nullpoints of $\beta_\infty = 0$ from the values of $a_{\infty 1}$ to $a_{\infty 7}$ in Table I, and draw the curves representing $\beta_2 = 0$ to $\beta_7 = 0$.

In studying these curves the following tendencies should be considered (see Table I and Fig. 4):

a) The nullpoints of these curves always approach the nullpoints of $\beta_\infty = 0$ as n increases.

b) The first $\bar{z}_{n1}$ and the corresponding minima values of β always increase numerically with n. $\lim\limits_{n=\infty} \overline{\beta}_{n1}$ must be examined.

c) The first maximum increases from $\beta_3 = 0$ to $\beta_4 = 0$ and afterwards decreases, apparently to a small positive limit for $\beta_\infty = 0$.

d) The other minima and maxima between two successive roots of $\beta_n = 0$ always decrease numerically as n increases, apparently toward a very small limit for $\beta_\infty = 0$.

e) In each curve, the maxima and minima for the first half of the arches retrograde and for the last half advance beyond the middle of the interval.

To discuss these tendencies and eventually to discover properties of $\beta_\infty = 0$, return to the formula

$$a_n = b\left((n-1)^2 - z\right) a_{n-1} - a_{n-2}.$$

Substituting $a_n = b^n (n-1)!^2 \beta_n$ gives

(31) $$\beta_n = \left(1 - \frac{z}{(n-1)^2}\right)\beta_{n-1} - \frac{1}{b^2 (n-1)^2 (n-2)^2}\beta_{n-2}.$$

First Minimum.

To explain tendency b) and to determine the limit of the first minimum, use equation (31) and Table IV. The first minimum evidently lies between $z_{\infty 1} = -16.9015$ and $z_{\infty 2} = -5.0524$, the first two roots of $\alpha_{\infty} = 0$ and of $\beta_{\infty} = 0$.

Beginning with $\beta_5 = 0$,

(32) $$\overline{\beta}_{51} = \left(1 + \frac{|\bar{z}_{51}|}{4^2}\right)\beta_{41} - \frac{1}{0.01 \times 4^2 \times 3^2}\beta_{31}, \text{ for } \bar{z}_{51} = -12.6,$$

(33) $$\overline{\beta}_{61} = \left(1 + \frac{|\bar{z}_{61}|}{5^2}\right)\beta_{51} - \frac{1}{0.01 \times 5^2 \times 4^2}\beta_{41}, \text{ for } \bar{z}_{61} = -13.0,$$

(34) $$\overline{\beta}_{71} = \left(1 + \frac{|\bar{z}_{71}|}{6^2}\right)\beta_{61} - \frac{1}{0.01 \times 6^2 \times 5^2}\beta_{51}, \text{ for } \bar{z}_{71} = -13.3,$$

(35) $$\overline{\beta}_{n1} = \left(1 + \frac{|\bar{z}_{n1}|}{(n-1)^2}\right)\beta_{n-1,1} - \frac{1}{0.01\,(n-1)^2(n-2)^2}\beta_{n-2}, \text{ for } z_{n1}.$$

Since $\overline{\beta}_{51}$, $\overline{\beta}_{61}$, $\overline{\beta}_{71}$ are known by computation to be negative and $|\overline{\beta}_{51}| < |\overline{\beta}_{61}| < |\overline{\beta}_{71}|$, it is evident from (35) that $|\overline{\beta}_{81}| < |\overline{\beta}_{91}| < \cdots < |\overline{\beta}_{\infty 1}|$, and that they are all negative, since the first term in the second member of (35) has the greater multiplier from $\overline{\beta}_{51}$ to $\overline{\beta}_{\infty 1}$, the last multiplier being a proper fraction and decreasing rapidly while the first multiplier $1 + \dfrac{|\bar{z}_{n1}|}{(n-1)^2}$ remains greater than unity.

The relation $|\overline{\beta}_{71}| < |\overline{\beta}_{81}| \cdots < |\overline{\beta}_{\infty 1}|$ is preserved when $\bar{z}_{\infty 1}$ is substituted for $\bar{z}_{71}$, $\bar{z}_{81}, \ldots$, and the inequalities are still greater, since less ordinates are substituted for maximum ordinates.

(36) However,

$$|\overline{\beta}_n| < \left|\left(1 + \frac{|\bar{z}_n|}{(n-1)^2}\right)\beta_{n-1}\right| \text{ and } |\beta_{n-1}| < \left|\left(1 + \frac{|\bar{z}_{n-1}|}{(n-2)^2}\right)\beta_{n-2}\right|,$$

since $\left|\dfrac{1}{0.01\,(n-1)^2\,(n-2)^2}\beta_{n-1}\right|$ has some small positive value while n is finite.

(37) Hence

$$|\overline{\beta}_{n1}| < \left(1 + \frac{|\bar{z}_{n1}|}{(n-1)^2}\right) \times \left(1 + \frac{|\bar{z}_{n1}|}{(n-2)^2}\right) \times \cdots \times \left(1 + \frac{|\bar{z}_{n1}|}{7^2}\right) \times |\beta_{71}|,$$

for $\bar{z}_{\infty 1}$. For $n = \infty$, $z_{n1} = -13.5$ approximately, the last figure being deter-

mined by the recursion formula. Computing $\beta_{71} = 1224.4$ for $\bar{z}_{\infty 1} = -13.5$ by Horner's method,

$$(38) \quad \lim_{n=\infty} |\bar{\beta}_{n1}| < \left(1 + \frac{13.5}{(n-1)^2}\right) \times \left(1 + \frac{13.5}{(n-2)^2}\right) \times \cdots \times \left(1 + \frac{13.5}{7^2}\right) \times 1224.4.$$

For an infinite product,

$$(39) \quad \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{k^2 \pi^2}\right) = \frac{e^z - e^{-z}}{2z}. \qquad \frac{z^2}{\pi^2} = 13.5, \ z = \pi\sqrt{13.5}.$$

$$(40) \quad |\bar{\beta}_{\infty 1}| < 1224.4 \, . \, \frac{e^{\pi\sqrt{13.5}} - e^{-\pi\sqrt{-13.5}}}{2\pi\sqrt{13.5}} \div \prod_{k=1}^{6} \left(1 + \frac{13.5}{k^2}\right).$$

$$(41) \quad \bar{\beta}_{\infty 1} = -8826.7.$$

The first minimum is therefore a finite number.

Second Minimum.

The second minimum first appears in $\beta_4 = 0$, and by Table IV,

$$(42) \quad |\bar{\beta}_{53}| < |\bar{\beta}_{43}|,$$

and both are negative.

By equation (31) and Table IV,

$$(43) \quad \bar{\beta}_{63} = \left(1 - \frac{9.5}{25}\right)\beta_{53} - \frac{1}{0.01 \, . \, 25 \, . \, 16}\beta_{43}, \text{ for } \bar{z}_{63} = +9.5;$$

$$(44) \quad \bar{\beta}_{73} = \left(1 - \frac{9.5}{36}\right)\beta_{63} - \frac{1}{0.01 \, . \, 36 \, . \, 25}\beta_{53}, \text{ for } \bar{z}_{73} = +9.5;$$

$$(45) \quad \bar{\beta}_{83} = \left(1 - \frac{9.4}{49}\right)\beta_{73} - \frac{1}{0.01 \, . \, 49 \, . \, 36}\beta_{63}, \text{ for } \bar{z}_{83} = +9.4.$$

By computing the coefficients of (43), (44), (45), it is found that β_{n3} continues negative and of decreasing numerical value. Since the second term of the second member is decreasing rapidly, on account of the factors $(n-1)^2$ $(n-2)^2$ in the denominator, toward the limit zero, and $1 - \frac{\bar{z}}{(n-1)^2}$ more slowly increases toward the limit unity, it is seen that $\bar{\beta}_{83}$ equals β_{73} multiplied by a positive proper fraction, plus a much smaller number. Thus $\bar{\beta}_{83}, \bar{\beta}_{93}, \ldots,$ $\bar{\beta}_{\infty 3}$ remain negative and $|\bar{\beta}_{83}| > |\bar{\beta}_{93}| \ldots > |\bar{\beta}_{\infty 3}|$.

$\lim_{n=\infty} \bar{\beta}_{n,3} = \bar{\beta}_{n-1,3}$, and $\bar{\beta}_{\infty}$ is a small negative number. By repeated use of (31),

$$(46) \quad \bar{\beta}_{\infty 3} = -27. \qquad \text{See Fig. 4.}$$

Third and Subsequent Minima.

The third and subsequent minima can be treated in the same manner. We observe that the first minimum in any set or range may be large as compared with the minima in the preceding sets. For example, $\overline{\beta}_{65}$, for $\bar{z}_{65} = + 29.8$, is -497.6. This irregularity is due to the fact that the first minimum in a set lies between the last two roots of an equation of an even degree; e. g., $\beta_6 = 0$. For $b < 1$, as in the case under consideration, the last root does not, in general, lie between $(n-1)^2$ and n^2 until $b(n-1) > 1$.*

In such a case $\bar{z}$ may be greater than $(n-1)^2$, as in $\bar{z}_{65}$. Hence

(47) $\overline{\beta}_{65} = \left(1 - \frac{\bar{z}_{65}}{(n-1)^2}\right)\beta_{55} - \frac{1}{0.01\,(n-1)^2\,(n-2)^2}\,\beta_{45}$ has irregularities. By computation,

$$\overline{\beta}_{65} = \left(1 - \frac{29.8}{25}\right)(-1673) - \tfrac{1}{4}(+3275.3), \text{ for } \bar{z}_{65} = +29.8. \tag{48}$$

$$\overline{\beta}_{65} = (1 - 1.192)(-1673) - \tfrac{1}{4}(+3275.3) = -497.6 \text{ approximately.} \tag{49}$$

The first parenthesis changes sign, since $\bar{z}_{65} > (n-1)^2$, and $\overline{\beta}_{65}$ therefore becomes a large positive number. However, the coefficient of the next term is comparatively large, so that the last term controls the sign of $\overline{\beta}_{65}$.

However,

$$\overline{\beta}_{75} = \left(1 - \frac{25.7}{36}\right)\beta_{65} - \frac{1}{9}\,\beta_{55} = -152.1, \text{ for } \bar{z}_{75} = 25.7. \tag{50}$$

Here and subsequently in $\overline{\beta}_{85}, \ldots, \overline{\beta}_{\infty 5}$ the parenthesis is positive, slowly approaching the limit unity, and both terms are negative. As the last coefficient decreases rapidly toward the limit zero, the values of $\overline{\beta}_{85}, \ldots, \overline{\beta}_{\infty 5}$ decrease numerically with increasing n and are always negative. Hence, as in the second minima,

$$\lim_{n=\infty} \overline{\beta}_{n5} = \overline{\beta}_{n-1,\,5} \tag{51}$$

and $\overline{\beta}_{\infty 5}$ is a very small negative number, and not zero.

The same argument applies to subsequent sets of minima values.

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* Heine's *Kugelfunktionen*, I, 407.

Maxima Values of β_∞.

The first maximum of each set comes from an equation of odd degree and may be large, as in $\bar{\beta}_{32}$, since $\bar{z}$ may be larger than $(n-1)^2$ for the last arch of the curve until $b(n-2)>1$ and thus make both terms of (31) positive. The first set of maxima values shows an increase in $\bar{\beta}_{42}$ on account of the small value of n and the factor $b^2\,[=0.01]$ in the denominator of the second term.

$$(52)\qquad \bar{\beta}_{32}=\left(1-\frac{11.7}{4}\right)\beta_{22}-\frac{1}{0.01\,.\,4\,.\,1}\,\beta_{12},\ \text{for } \bar{z}_{32}=+11.7,$$
$$=(-1.9)(-38)-25\,(-5.85)=+72.2+146.25=+218.45.$$

$$(53)\qquad \bar{\beta}_{42}=\left(1-\frac{4.4}{9}\right)\beta_{32}-\frac{1}{0.01\,.\,9\,.\,4}\,\beta_{22},\ \text{for } \bar{z}_{42}=+4.4,$$
$$=(+0.51)(+90)-2.8\,(-87)=+4.59+243.6=+289.5.$$

$$(54)\qquad \bar{\beta}_{52}=\left(1-\frac{0.7}{16}\right)\beta_{42}-\frac{1}{0.01\,.\,16\,.\,9}\,\beta_{32},\ \text{for } \bar{z}_{52}=+0.7,$$
$$=(+0.956)(+220)-0.694\,(-60)=+251.9+41.6=+251.9.$$

The subsequent maxima $\bar{\beta}_{62},\ \ldots,\ \bar{\beta}_{\infty 2}$ will decrease and remain positive, since $\dfrac{1}{0.01\,(n-1)^2\,(n-2)^2}$ decreases rapidly and $1-\dfrac{z}{(n-1)^2}$ remains approximately $+1$.

Hence $\lim\limits_{n=\infty}\bar{\beta}_{n2}$ is a small positive number. From Table IV and equation (31), the approximate limit is $\bar{\beta}_{\infty 2}=140$.

By the same argument, the subsequent maxima may be shown to be finite and not zero.

Maxima and Minima Values of a_∞.

$$\begin{aligned}\text{Max. } a_n &= b^n\,(n-1)\,!^2\ .\ \text{Max. } \beta_n,\\ \text{Min. } a_n &= b^n\,(n-1)\,!^2\ .\ \text{Min. } \beta_n.\end{aligned}\qquad (9)$$

Since the maxima and minima values of β_∞, for $b=0.1$, $b=10$ and $b=100$, are finite and not zero,

$$\begin{aligned}\text{Max. } a_\infty &= \infty\ .\ \text{Max. } \beta_\infty=\infty,\\ \text{Min. } a_\infty &= \infty\ .\ \text{Min. } \beta_\infty=\infty.\end{aligned}$$

Considering the factor b^n in these results, it is evident that the curves representing $a_n=0$ will be more nearly perpendicular to the z-axis for $b=10$ and $b=100$ than for $b=0.1$. This conclusion is confirmed by Tables I, II, III and Fig. 5.

Considering the variations in a_{ni} in these tables for z_{ni} to four decimal places, it will be noticed that, in general, a_{ni} becomes greater as n increases and the pitch of the curves are greater at the nullpoints. Moreover, a_{ni} is greater in the vicinity of the nullpoint on the side nearest the maximum or minimum point of the arch, due allowance being made for the fact that the next figure of z_{ni} in the table may be very large or very small in the values compared. It should also be noted that a point of inflexion exists in every arch, near the nullpoint which is more remote from the maximum or minimum point of the arch.

CONVERGENCE OF THE SERIES

$$E(\phi) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\phi. \tag{55}$$

I. *Argument* $b = 0.1$.

The computation of $a_1, a_2, \ldots, a_n$ for a finite n forms a basis for more general conclusions. The method employed in determining the roots of $a_\infty = 0$ and the difficulties encountered in these computations will be illustrated by finding the fifth decimal figure of $z_{\infty 3}$, for argument $b = 0.1$. Having already $z_{\infty 3} = 5.5813$, we compute a_6 and a_7 from (6) for $z = 5.58138$ by Horner's method, using the remainder to twenty-one decimal places for the determination of a_6 and a_7. Then by the recursion formula we compute $a_8, a_9, \ldots\ldots$

$$a_{n+2} = b\left((n+1)^2 - z\right) a_{n+1} - a_n. \tag{56}$$

(57) For $z = 5.58138$,
$$\begin{aligned} a_6 &= +\ 0.22846013 \\ a_7 &= +\ 0.05475009 \\ a_8 &= +\ 0.00905173 \\ a_9 &= -\ 0.00187210 \end{aligned}$$

Evidently 5.58138 is too large for $z_{\infty 3}$, since a_9 is negative and, by (56), $a_{10}, a_{11}, \ldots, a_\infty$ would all be negative and would increase numerically with n, since the coefficient of a_{n+1} would be greater than 2 and would increase indefinitely. Hence $a_{\infty 3}$ would not be zero.

(58) For $z = 5.58137$,
$$\begin{aligned} a_6 &= +\ 0.228465748 \\ a_7 &= +\ 0.054764586 \\ a_8 &= +\ 0.009314650 \\ a_9 &= -\ 0.000442830 \end{aligned}$$

This value of z is still too great, but a_9 is much smaller numerically.

(59) For $z = 5.58135$,
$$\begin{aligned} a_6 &= +\,0.228476975 \\ a_7 &= +\,0.054793576 \\ a_8 &= +\,0.009429440 \\ a_9 &= +\,0.000197600 \\ a_{10} &= -\,0.007939170 \end{aligned}$$

Here a_9 is positive but a_{10} is negative, and $z = 5.58135$ locates a point in the z-axis between the nullpoints of $a_9 = 0$ and $a_{10} = 0$.

Computing $a_1, a_2, \ldots, a_7$ by Horner's method,

(60) For $z = 5.58134$,
$$\begin{aligned} a_1 &= -\,0.279067000 \\ a_2 &= -\,0.872149970 \\ a_3 &= +\,0.416988556 \\ a_4 &= +\,1.014702419 \\ a_5 &= +\,0.640200395 \\ a_6 &= +\,0.228482548 \\ a_7 &= +\,0.054808073 \\ a_8 &= +\,0.009687228 \\ a_9 &= +\,0.001783414 \\ a_{10} &= +\,0.003763039 \end{aligned}$$

It is evident that 5.58134 is the value of $z_{\infty 3}$, correct to five decimal places; since $a_{10} < a_{11} < a_{12} \ldots < a_{\infty}$ by (56), and all the curves $a_{10} = 0$ to $a_{\infty} = 0$ are still above the z-axis for $z = 5.58134$, but below the z-axis for $z = 5.58135$.

Substituting (60) in (55),

(61)
$$\begin{aligned} E(\phi) = 0.5 &- 0.2790 \cos 2\phi - 0.8721 \cos 4\phi + 0.4169 \cos 6\phi \\ &+ 1.0147 \cos 8\phi + 0.6402 \cos 10\phi + 0.2284 \cos 12\phi \\ &+ 0.0547 \cos 14\phi + 0.0096 \cos 16\phi + 0.0017 \cos 18\phi \\ &+ 0.0037 \cos 20\phi + \ldots\ldots \end{aligned}$$

The accuracy of these coefficients is proved by the recursion formula (56).

Convergence of the Series.

This series is computed for a root $z_{\infty 3}$, accurate to the fifth decimal place. By a careful consideration of the last remainders in Horner's process used in computing $a_1, a_2, \ldots, a_6$ it is evident that the first three decimal figures in these coefficients will never change, if an infinite number of figures of $z_{\infty 3}$ are computed and substituted in $E(\phi)$, since an increase of a whole unit in the sixth

figure of z does not cause a change in the first three decimal places of a_1, a_2, $\ldots$, a_6.

Calling the finite sum of the first four terms F, we write,

$$(62)\qquad |E(\phi)| < |F \pm (1.0147 + 0.6402 + 0.2284 + 0.0548 + 0.0096 + \ldots)|,$$

substituting maximum values for $\cos 8\phi$, $\cos 10\phi$, $\ldots$, and using the $+$ sign when F and the following series have like signs, and the $-$ sign when they have opposite signs.

We must now define a root of $a_\infty = 0$ more carefully.

(63) *Definition of a Root of* $a_\infty = 0$.

From a certain a_n onward indefinitely, for an exact root of $a_\infty = 0$, a_{n+1} is less than a_n and has the same sign to $n = \infty$.

Otherwise one of the following relations must exist:

(64) 1) $a_{n+1} > a_n$, with the same sign.

(65) 2) a_{n+1} and a_n differ in sign.

(66) 3) $a_{n+1} < a_n$ for one or more terms and then $a_{n+1} > a_n$.

Supposition (64) can not be true for $z_{\infty 1}$ a root of $a_\infty = 0$, since a_{n+2}, a_{n+3}, $\ldots$, a_∞ would form an increasing series, as is shown by the formula

$$(67)\qquad a_{n+2} = b\,((n+1)^2 - z)\,a_{n+1} - a_n.$$

When $b\,((n+1)^2 - z)$ becomes greater than 2 with increasing n and $a_{n+1} > a_n$, a_{n+2} must be greater than a_{n+1}. Hence $\lim\limits_{n=\infty} a_n$ would not be zero.

The second supposition (65) is false for a root of $a_\infty = 0$, because (67) shows that, when a_{n+1} and a_n have opposite signs and the coefficient of a_{n+1} becomes and remains positive with increasing n, a_{n+2} must, under condition (65), have a larger absolute value than a_{n+1}. Hence $\lim\limits_{n=\infty} a_n$ is not zero.

The third supposition (66) reduces to (64) or to (65) and therefore can not be true.

Hence (63) defines a root of $a_\infty = 0$.

This definition gives the following rule:

(68) *Rule for Computing Roots of* $a_\infty = 0$.

Find the successive figures of positive roots of $a_\infty = 0$ as great as possible, so that a_6, a_7, a_8, $\ldots$, a_∞ shall have the same signs.

For negative roots, each succeeding figure of $z_{\infty 1}$ is one less than the least figure that gives a permanence in the signs of a_6, a_7, a_8, ..., a_∞.

For a root of $a_\infty = 0$, as above defined, (67) gives

$$a_{n+2} = b\,((n+1)^2 - z)\,a_{n+1} - a_n < a_{n+1},$$

$$a_{n+1} < \frac{a_n}{b\,((n+1)^2 - z) - 1}. \tag{69}$$

From a certain n onward to $n = \infty$,

$$a_{n+1} < \tfrac{1}{2}\,a_n. \tag{70}$$

Therefore the last series in (62) may be written

$$1.0147 + 0.6402 + 0.2284 + \ldots\ldots < 1.0147 + 0.6402\,(1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots\ldots), \tag{71}$$

which is clearly a converging series.

Hence from (62) and (71), when $z_{\infty 3}$ is an exact root of $a_\infty = 0$,

(72) $E(\phi) = \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\phi$ is finite and the series is convergent.

II. *Argument* $b = 10$.

$$E(\phi) = \tfrac{1}{2}\,a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\phi. \tag{73}$$

The computation of the values of a_n with sufficient accuracy to show that a_n converges toward the limit zero and that the series representing $E(\phi)$ is convergent when n increases without limit, involves difficulties due to the rapidity with which the curves $a_n = 0$ approach perpendicularity to the z-axis at the nullpoints.

For example, we compute the third root of $a_7 = 0$ by Horner's method, carrying the remainders to eighteen decimal places, and find the values of a_6 and a_7 for this root as we obtain the successive approximations. These computations are continued until the corresponding values of a_{63} and a_{73} give a negative value for a_{83} by the recursion formula.

We find $z_{73} = 4.00133 +$, and observe from the recursion formula that

$$a_8 = 10\,(49 - 4.00133)\,a_7 - a_6 = 450\,a_7 - a_6, \text{ approximately.}$$

Evidently a_8 will not become negative until $450\,a_7$ is less than a_6. When these values of a_6 and a_7 are found, $z_{\infty 3}$ can be found as follows:

	z_{73}	a_{73}	a_{63}
(74)	4	3232143580.000000	10100579.000000
	0	"	"
	0	"	"
	1	814029735.729124	2543960.282771
	3	88386514.036511	276223.188814
	3	15816887.536327	49430.563659
	6	1302846.520815	4076.625965
	5	93341.361791	291.709237
	3	20771.043732	64.914147
	8	1418.958754	4.435519
	5	208.453440	0.655601
	8	15.932590	0.050815
	6	1.418526	0.005456
	5	0.209021	0.001676
	8	0.015500	0.001071
	6	0.000956	0.001026
	3	0.000231	0.001023
	9	0 000013	0.001023
	5	0.000001	0.001023

Substituting the last results in (67), using z_{73} to eighteen decimal places,

(75) $$a_8 = 10\,(49 - 4.001336538586586395)\,.\,0.0000012 - 0.0010230 \\ = -\,0.000490.$$

Since a_8 is negative, z is too large by (63).

Taking z to seventeen decimal places,

(76) $$a_8 = 10\,(49 - 4.00133653858658639)\,.\,0.0000133 - 0.0010231 \\ = +\,0.0049617.$$

Since a_8 is here positive and in the former case negative, $z_{\infty 3}$ lies between the two values taken.

$$z_{\infty 3} = 4.00133653858658639 + .$$

Subsequent figures may be found as in (57) to (60).

Hence,

$$(77)\quad E(\phi) = 0.5 + \sum_{n=1}^{5} a_n \cos 2n\phi + 0.0010231 \cos 12\phi + 0.0000013 \cos 14\phi + \ldots\ldots$$

The trial divisor in Horner's process contained twenty figures when the contract method was used in determining the last fifteen figures of z. To secure values for a_1 to a_8 that will satisfy the recursion formula, twenty-one decimal places should be used in all remainders and the contract method should be employed later in the work.

It will be noted that even with seventeen decimal places of $z_{\infty 3}$ accurately computed, a_{83} is greater than a_{73}. It is evidently practically impossible to find a value for $z_{\infty 3}$ so near the exact root that a_{73}, a_{83}, ..., $a_{\infty 3}$ shall form a rapidly decreasing series. Heine* remarks that when such a value is found,

$$(78)\qquad a_{n+1} < \frac{a_n}{b\,[(n+1)^2 - z] - 1},$$

and therefore with increasing n, a_n decreases rapidly to the limit zero Practically each succeeding figure of $z_{\infty i}$ must be found as in (57) to (60).

For argument $b = 100$, similar treatment will give like conclusions. As the pitch of the curves is greater at the nullpoints, more decimal places will be required and greater difficulties in computation will be encountered.

The Best Values of b and z.

To find a finite number of terms of

$$(79)\qquad E(\phi) = \tfrac{1}{2} a_0 + a_1 \cos 2\phi + a_2 \cos 4\phi + \ldots.$$

that will give a good approximate solution of the equation

$$(80)\qquad \frac{d^2 E(\phi)}{d\phi^2} + \left(\frac{8}{b} \cos 2\phi + 4z\right) E(\phi) = 0,$$

it is evident from the preceding discussion that $z_{\infty i}$ must be computed to four or more decimal places when $b = 0.1$ and to many more places when b is a larger number. It will be seen that small values of b are desirable; since the curves representing $a_n = 0$ have a steeper pitch for larger values of b, and consequently a larger number of decimal figures of $z_{\infty i}$ must be computed in the latter case to obtain a_n and a_{n+1} small enough to satisfy the necessary condition given in

* *Kugelfunktionen*, I, 411.

equation (4). Fairly good values of the coefficients of our series for $b = 0.1$ have been found in computing $z_{\infty 1}$ to $z_{\infty 7}$.

$$(81)\quad z_{\infty 1} = -16.9015. \quad E_1 = 0.5 + \sum_{n=1}^{5} a_n \cos 2n\phi + 0.004242 \cos 12\phi + 0.003259 \cos 14\phi + \ldots\ldots$$

$$z_{\infty 2} = -\ \ 5.0524. \quad E_2 = 0.5 + \sum_{n=1}^{5} a_n \cos 2n\phi - 0.055101 \cos 12\phi - 0.013109 \cos 14\phi - 0.002469 \cos 16\phi - 0.001497 \cos 18\phi - \ldots\ldots$$

$$z_{\infty 3} = +\ \ 5.5813. \quad E_3 = 0.5 + \sum_{n=1}^{5} a_n \cos 2n\phi + 0.228301 \cos 12\phi + 0.054866 \cos 14\phi + 0.009922 \cos 16\phi + 0.003074 \cos 18\phi + \ldots\ldots$$

$$z_{\infty 4} = +\ 14.5616. \quad E_4 = 0.5 + \sum_{n=1}^{5} a_n \cos 2n\phi - 0.434188 \cos 12\phi - 0.131937 \cos 14\phi - 0.030213 \cos 16\phi - \ldots\ldots$$

$$z_{\infty 5} = +\ 20.4705. \quad E_5 = 0.5 + \sum_{n=1}^{5} a_n \cos 2n\phi + 0.627187 \cos 12\phi + 0.233685 \cos 14\phi + 0.039501 \cos 16\phi + 0.005334 \cos 18\phi + 0.002985 \cos 20\phi + \ldots\ldots$$

$$z_{\infty 6} = +\ 27.1931. \quad E_6 = 0.5 + \sum_{n=1}^{5} a_n \cos 2n\phi - 5.360337 \cos 12\phi - 2.830106 \cos 14\phi - 0.811261 \cos 16\phi - 0.155902 \cos 18\phi - 0.027635 \cos 20\phi - \ldots\ldots$$

$$z_{\infty 7} = +\ 37.3476. \quad E_7 = 0.5 + \sum_{n=1}^{5} a_n \cos 2n\phi + 69.726061 \cos 12\phi + 51.778903 \cos 14\phi + 28.841160 \cos 16\phi + 6.882976 \cos 18\phi + 1.142740 \cos 20\phi + 0.166280 \cos 22\phi + \ldots\ldots$$

It will be noticed in these equations, that, in general, the smaller the roots are algebraically the better are the coefficients obtained for an approximate solution in the form of a finite number of terms of the infinite series. From Fig. 5, we should expect the best approximations for E_5 and E_4; and this is doubtless in general true, since the maxima and minima are here less numerically and the slopes of the curves at the nullpoints are not so great, but the possible difference in the fifth and subsequent decimal places of $z_{\infty 1}, \ldots, z_{\infty 7}$ make it impossible to determine this fact definitely without further calculations.

To obtain, for $b = 10$, as good approximate results as the above, the values of $z_{\infty 1}, \ldots, z_{\infty 7}$ must be computed to eighteen or more decimal places and, for $b = 100$, to many more places.

As a general conclusion, it is evident that small values of b and z are the best values.

General Proof of Convergence.

Granting that $a_\infty = f(b, z) = 0$ is an equation of an infinite degree in the general form of equations (5) and that the coefficients of the series in (2) satisfy the recursion formula (3) for exact roots of $a_\infty = 0$, we have shown that these roots can be computed to any desired degree of accuracy by rule (68). We must now show that the series in the solution $E(\phi) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\phi$ is convergent for values of a_n computed for $z_{\infty\, i}$, any exact root of $a_\infty = 0$.

Beginning with a certain a_m, definition (63) gives

$$a_{m+1} < \frac{a_m}{b\left((m+1)^2 - z\right) - 1}, \tag{69}$$

$$a_{m+1} < \tfrac{1}{2} a_m. \tag{70}$$

(82) $$E(\phi) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\phi$$
$$= \left(\tfrac{1}{2} a_0 + a_1 \cos 2\phi + \ldots + a_{m-1} \cos 2(m-1)\phi\right)$$
$$+ \left(a_m \cos 2m\phi + \ldots + a_\infty \cos \infty\phi\right).$$

If a_m is the first term of the decreasing series that characterizes a root of $a_\infty = 0$ in definition (63) and satisfies relation (70),

$$|E(\phi)| < \left|\left(\tfrac{1}{2} a_0 + a_1 \cos 2\phi + \ldots + a_{m-1} \cos 2(m-1)\phi\right) \pm \left(a_m + a_{m+1} + \ldots + a_\infty\right)\right|,$$

substituting maximum values for $\cos 2m\phi, \ldots$, and using the plus sign when the two parentheses are alike in sign, and the minus sign when they are of opposite sign.

$$|a_m + a_{m+1} + \ldots + a_\infty| < |a_m(1 + \tfrac{1}{2} + \tfrac{1}{4} + \ldots)| = |2a_m|. \tag{70}$$

The quantities $a_1, a_2, \ldots, a_m$ are finite, since a_n is a rational, integral function of z with finite coefficients; hence $a_1, a_2, \ldots, a_m$ must be finite for finite values of z.

Hence,

$$|E(\phi)| < |(\text{Finite Sum} \pm 2a_m)|, \text{ and}$$

$E(\phi) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\phi$ is a convergent series and a solution of

$$\frac{d^2E}{d\phi^2} + \left(\frac{8}{b} \cos 2\phi + 4z\right) E = 0.$$

TABLE I.

$b = 0.1$

Z_{11}	+ 0.0000	A_{11}	+0.00000
Z_{21}	−13.6509	A_{21}	−0.00000
Z_{22}	+14.6509	A_{22}	−0.00000
Z_{31}	−16.2479	A_{31}	−0.00002
Z_{32}	+ 2.6470	A_{32}	−0.00000
Z_{33}	+18.6009	A_{33}	+0.00002
Z_{41}	−16.8064	A_{41}	−0.00000
Z_{42}	− 2.8853	A_{42}	+0.00002
Z_{43}	+12.3932	A_{43}	+0.00000
Z_{44}	+21.2985	A_{44}	−0.00001
Z_{51}	−16.8934	A_{51}	−0.00015
Z_{52}	− 4.6582	A_{52}	+0.00003
Z_{53}	+ 7.7970	A_{53}	+0.00001
Z_{54}	+18.0760	A_{54}	−0.00002
Z_{55}	+25.6784	A_{55}	+0.00034
Z_{61}	−16.9011	A_{61}	−0.00013
Z_{62}	− 5.0133	A_{62}	+0.00008
Z_{63}	+ 5.9979	A_{63}	+0.00004
Z_{64}	+15.7525	A_{64}	−0.00002
Z_{65}	+21.9884	A_{65}	+0.00003
Z_{66}	+33.1754	A_{66}	−0.00074
Z_{71}	−16.9015	A_{71}	−0.00109
Z_{72}	− 5.0500	A_{72}	+0.00008
Z_{73}	+ 5.6193	A_{73}	+0.00000
Z_{74}	+14.7525	A_{74}	−0.00004
Z_{75}	+20.7688	A_{75}	+0.00006
Z_{76}	+28.6580	A_{76}	−0.00021
Z_{77}	+43.1526	A_{77}	+0.00348
$Z_{\infty 1}$	−16.9015		
$Z_{\infty 2}$	− 5.0524		
$Z_{\infty 3}$	+ 5.5813		
$Z_{\infty 4}$	+14.5616		
$Z_{\infty 5}$	+20.4705		
$Z_{\infty 6}$	+27.1931		
$Z_{\infty 7}$	+37.3476		

TABLE II.

$b = 10$

Z_{11}	+ 0.0000	A_{11}	+ 0.00000
			$\times 10^{1}$
Z_{21}	− 0.0196	A_{21}	− 0.00002
Z_{22}	+ 1.0196	A_{22}	− 0.00002
			$\times 10^{2}$
Z_{31}	− 0.0196	A_{31}	− 0.00013
Z_{32}	+ 1.0163	A_{32}	− 0.00006
Z_{33}	+ 4.0033	A_{33}	+ 0.00071
			$\times 10^{3}$
Z_{41}	− 0.0196	A_{41}	− 0.00112
Z_{42}	+ 1.0163	A_{42}	− 0.00032
Z_{43}	+ 4.0013	A_{43}	+ 0.00110
Z_{44}	+ 9.0019	A_{44}	− 0.01295
			$\times 10^{4}$
Z_{51}	− 0.0196	A_{51}	− 0.01875
Z_{52}	+ 1.0163	A_{52}	− 0.00476
Z_{53}	+ 4.0013	A_{53}	+ 0.01315
Z_{54}	+ 9.0005	A_{54}	− 0.09021
Z_{55}	+16.0014	A_{55}	+ 0.28200
			$\times 10^{5}$
Z_{61}	− 0.0196	A_{61}	− 0.52699
Z_{62}	+ 1.0163	A_{62}	− 0.08015
Z_{63}	+ 4.0013	A_{63}	+ 0.27626
Z_{64}	+ 9.0005	A_{64}	− 1.44069
Z_{65}	+16.0003	A_{65}	+ 1.59149
Z_{66}	+25.0011	A_{66}	−10.02866
			$\times 10^{6}$
Z_{71}	− 0.0196	A_{71}	− 16.90346
Z_{72}	+ 1.0163	A_{72}	− 3.86381
Z_{73}	+ 4.0013	A_{73}	+ 3.33886
Z_{74}	+ 9.0005	A_{74}	− 7.26793
Z_{75}	+16.0003	A_{75}	+ 31.69049
Z_{76}	+25.0002	A_{76}	−196.55479
Z_{77}	+36.0009	A_{77}	+1084.83159
			$\times 10^{7}$
$Z_{\infty 1}$	− 0.0196		
$Z_{\infty 2}$	+ 1.0163		
$Z_{\infty 3}$	+ 4.0013		
$Z_{\infty 4}$	+ 9.0005		
$Z_{\infty 5}$	+16.0003		
$Z_{\infty 6}$	+25.0002		
$Z_{\infty 7}$	+36.0001		

TABLE III.

$b = 100$

Z_{11}	+ 0.0000	A_{11}	+ 0.00000
			$\times 10^{2}$
Z_{21}	− 0.0001	A_{21}	− 0.00004
Z_{22}	+ 1.0001	A_{22}	− 0.00004
			$\times 10^{4}$
Z_{31}	− 0.0001	A_{31}	− 0.00019
Z_{32}	+ 1.0001	A_{32}	− 0.00009
Z_{33}	+ 4.0000	A_{33}	+ 0.00020
			$\times 10^{6}$
Z_{41}	− 0.0001	A_{41}	− 0.00179
Z_{42}	+ 1.0001	A_{42}	− 0.00079
Z_{43}	+ 4.0000	A_{43}	+ 0.00040
Z_{44}	+ 9.0000	A_{44}	− 0.00359
			$\times 10^{8}$
Z_{51}	− 0.0001	A_{51}	− 0.02879
Z_{52}	+ 1.0001	A_{52}	− 0.01199
Z_{53}	+ 4.0000	A_{53}	+ 0.00480
Z_{54}	+ 9.0000	A_{54}	− 0.00719
Z_{55}	+16.0000	A_{55}	+ 0.14400
			$\times 10^{10}$
Z_{61}	− 0.0001	A_{61}	− 0.71997
Z_{62}	+ 1.0001	A_{62}	− 0.28791
Z_{63}	+ 4.0000	A_{63}	+ 0.10080
Z_{64}	+ 9.0000	A_{64}	− 0.11520
Z_{65}	+16.0000	A_{65}	+ 0.28798
Z_{66}	+25.0000	A_{66}	− 8.83001
			$\times 10^{12}$
Z_{71}	− 0.0001	A_{71}	−25.91922
Z_{72}	+ 1.0001	A_{72}	−10.07714
Z_{73}	+ 4.0000	A_{73}	+ 3.22567
Z_{74}	+ 9.0000	A_{74}	− 3.11034
Z_{75}	+16.0000	A_{75}	+ 5.76028
Z_{76}	+25.0000	A_{76}	−20.16004
Z_{77}	+36.0000	A_{77}	+1088.63940
			$\times 10^{14}$
$Z_{\infty 1}$	− 0.0001		
$Z_{\infty 2}$	+ 1.0000		
$Z_{\infty 3}$	+ 4.0000		
$Z_{\infty 4}$	+ 9.0000		
$Z_{\infty 5}$	+16.0000		
$Z_{\infty 6}$	+25.0000		
$Z_{\infty 7}$	+36.0000		

TABLE IV.

$b = 0.1$		$+\bar{\beta} = $ Max. $-\bar{\beta} = $ Min.		
	$a_n = 0,\ \beta_n = 0$	$\bar{z}$	$\bar{\beta}$	$\bar{a}$
Z_{21}	-13.6509	$+0.5$	-100.1	-1.00100
Z_{22}	$+14.6509$			
Z_{31}	-16.2479	-8.4	-292.6	-1.17040
Z_{32}	$+2.6470$	$+11.7$	$+218.2$	$+0.87280$
Z_{33}	$+18.6009$			
Z_{41}	-16.8064	-11.4	-497.4	-1.79071
Z_{42}	-2.8853	$+4.4$	$+289.6$	$+1.04256$
Z_{43}	$+12.3932$	$+17.8$	-188.0	-0.67680
Z_{44}	$+21.2985$			
Z_{51}	-16.8934	-12.6	-708.8	-4.08311
Z_{52}	-4.6582	$+0.7$	$+252.4$	$+1.45433$
Z_{53}	$+7.7970$	$+12.9$	-156.7	-0.81259
Z_{54}	$+18.0760$	$+22.7$	$+192.9$	$+1.11110$
Z_{55}	$+25.6784$			
Z_{61}	-16.9010	-13.0	-1137.2	-16.37613
Z_{62}	-5.0133	-0.5	$+205.5$	$+2.96017$
Z_{63}	$+5.9979$	$+9.5$	-86.1	-1.24016
Z_{64}	$+15.7525$	$+19.2$	$+49.1$	$+0.70733$
Z_{65}	$+21.9884$	$+29.8$	-497.6	-7.16590
Z_{66}	$+33.1754$			
Z_{71}	-16.9015	-13.3	-1227.4	-63.63132
Z_{72}	-5.0500	-1.0	$+165.9$	$+8.60357$
Z_{73}	$+5.6193$	$+9.1$	-47.0	-2.44122
Z_{74}	$+14.7525$	$+17.8$	$+23.2$	$+1.20588$
Z_{75}	$+20.7688$	$+25.7$	-152.1	-7.87878
Z_{76}	$+28.6580$	$+39.5$	$+1431.7$	$+74.22189$
Z_{77}	$+43.1526$			

$a_n = b^n\,(n-1)\,!^2\,\beta_n$

TABLE V.

$b = 10$		$+\bar{\beta} = $ Max. $-\bar{\beta} = $ Min.		
	$a_n = 0,\ \beta_n = 0$	$\bar{z}$	$\bar{\beta}$	$\bar{a}$
Z_{21}	-0.0196	$+0.5$	-0.13	-0.00130×10^4
Z_{22}	$+1.0196$			
Z_{31}	-0.0196	$+0.4$	-0.11	-0.00048×10^6
Z_{32}	$+1.0163$	$+2.8$	$+0.75$	$+0.00300 \times 10^6$
Z_{33}	$+4.0033$			
Z_{41}	-0.0196	$+0.4$	-0.10	-0.00036×10^8
Z_{42}	$+1.0163$	$+2.7$	$+0.52$	$+0.00187 \times 10^8$
Z_{43}	$+4.0013$	$+7.7$	-3.43	-0.01235×10^8
Z_{44}	$+9.0019$			
Z_{51}	-0.0196	$+0.4$	-0.10	-0.00062×10^{10}
Z_{52}	$+1.0163$	$+2.7$	$+0.45$	$+0.00261 \times 10^{10}$
Z_{53}	$+4.0013$	$+7.0$	-1.96	-0.01134×10^{10}
Z_{54}	$+9.0005$	$+13.7$	$+15.84$	$+0.09129 \times 10^{10}$
Z_{55}	$+16.0014$			
Z_{61}	-0.0196	$+0.4$	-0.10	-0.00153×10^{12}
Z_{62}	$+1.0163$	$+2.6$	$+0.38$	$+0.00558 \times 10^{12}$
Z_{63}	$+4.0013$	$+6.9$	-1.41	-0.02041×10^{12}
Z_{64}	$+9.0005$	$+13.4$	$+7.20$	$+0.10364 \times 10^{12}$
Z_{65}	$+16.0003$	$+22.3$	-69.33	-0.99847×10^{12}
Z_{66}	$+25.0011$			
Z_{71}	-0.0196	$+0.4$	-0.10	-0.00547×10^{14}
Z_{72}	$+1.0163$	$+2.6$	$+0.35$	$+0.01850 \times 10^{14}$
Z_{73}	$+4.0013$	$+6.9$	-1.14	-0.05941×10^{14}
Z_{74}	$+9.0005$	$+13.2$	$+4.51$	$+0.23393 \times 10^{14}$
Z_{75}	$+16.0003$	$+21.9$	-26.29	-1.36291×10^{14}
Z_{76}	$+25.0002$	$+34.0$	$+1551.99$	$+80.45630 \times 10^{14}$
Z_{77}	$+36.0009$			

$\bar{a} = b^n\,(n-1)\,!^2\,\bar{\beta}$

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